Stochastic programs for identifying critical structural collapse mechanisms

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An important task in structural analysis is the identification of critical structural collapse mechanisms, that is, given a structural design and loading configuration, determine which failure mechanisms will occur first. In this work we extend traditional deterministic optimization models for failure mode identification to stochastic forms by considering external loads and structural member plastic moment capacities as correlated random variables. A hybrid model is first developed that contains both chance constrained programming and stochastic linear programming features. A purely chance constrained model is then described. Both models represent nonconvex programming problems. Computational experience with the models is described through an application to the analysis of a portal frame.

Keywords: stochastic optimization, nonlinear optimization, structural analysis

Introduction

The estimation of critical collapse load factors is a long-standing problem in structural analysis. Referring to the portal frame in *Figure 1*, the intent is to determine which kinematically admissible (i.e., geometrically consistent) failure mechanism (beam failure, sway failure, or combined beam and sway failure through joint rotation) will occur first in a given structure that is subject to a specified external loading condition. The failure mechanism so identified possesses a minimum ratio of internal to external work. This ratio is the so-called critical collapse load factor.

Optimization-based approaches to identify critical collapse mechanisms have been used by Dorn and Greenberg,[1] Charnes et al.,[2] Grierson and Gladwell,[3] Cohn et al.,[4] Moses,[5] Augusti et al.,[6] Thoft-Christensen and Murotsu,[7] and Nafday et al.[8] As originally formulated, the problem is

$$\textit{Minimize:} \quad \frac{\sum_{j=1}^{s} M_{pj}|\theta_j|}{\sum_{i=1}^{m} t_i e_i} \tag{1}$$

$$\textit{subject to:} \quad \theta_j - \sum_{i=1}^{m} t_i \theta_{ij} = 0 \qquad j = 1, 2, \ldots, s \tag{2}$$

where the M_{pj} are the plastic moment capacities at critical sections j, θ_j is the (net) plastic rotation, θ_{ij} is the plastic rotation for section j and elementary failure mechanism i, e_i is the external work term associated with elementary failure mechanisms i, and t_i is a linear multiplier that allocates external work across failure modes (i.e., *combined* failure mechanisms exist that are linear combinations of the elementary mechanisms).

Traditionally, the computational complications associated with the nonlinear objective function (equation (1)) have been avoided by normalization, that is, arbitrarily setting the external work ($\sum_{i=1}^{m} t_i e_i$) to unity, thus formulating the denominator of equation (1) as an equality constraint. In addition, all decision variables in the linear program (LP) are restricted to be nonnegative. The plastic rotations can, however, be positive or negative (clockwise or counterclockwise), as can the linear multipliers. Therefore we require the transformations (see Ref. 9, p. 168)

$$\theta_j = \theta_j^+ - \theta_j^- \tag{3}$$

$$t_i = t_i^+ - t_i^- \tag{4}$$

which lead to the separable linear programming (LP) model

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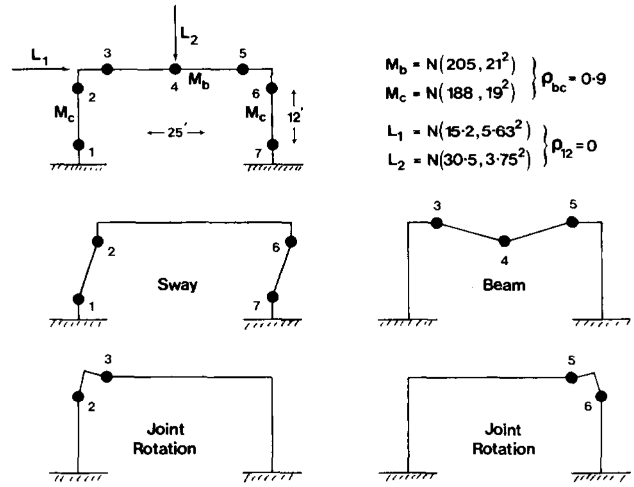

Figure 1. Portal frame: elementary collapse mechanisms

$$\text{Minimize:} \quad \sum_{j=1}^{s} M_{pj}(\theta_j^+ + \theta_j^-) \tag{5}$$

$$\text{subject to:} \quad \theta_j^+ - \theta_j^- - \sum_{i=1}^{m} (t_i^+ - t_i^-)\,\theta_{ij} = 0 \tag{6}$$

$$\sum_{i=1}^{m} (t_i^+ - t_i^-)\,e_i = 1 \tag{7}$$

Solving the model yields the minimum collapse load factor (equation (5)) and the failure mode associated with that factor.

Fractional programming

Before proceeding to the stochastic versions of the models, we note that while the aforementioned normalization seems straightforward, the formal equivalence of the original problem (equations (1)–(2)) and the normalized LP (equations (5)–(7)) has apparently not been established. These models, however, are indeed equivalent from a mathematical programming perspective, which can be shown by referring to certain well-established results from *fractional programming*.[10] The traditional fractional program

$$\text{Minimize:} \quad \frac{\mathbf{p}^T\mathbf{x} + \xi}{\mathbf{q}^T\mathbf{x} + \psi} \tag{8}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{9}$$

$$\mathbf{x} \geq \mathbf{O} \tag{10}$$

can be replaced by (at most) two ordinary linear programs that differ from each other by a change in sign in the objective function and in a constraint. If the denominator of equation (8) is positive (as it is in our problem, since it represents work), then an optimal solution to the original fractional program can be found by solving

$$\text{Minimize:} \quad \mathbf{p}^T\mathbf{y} + \xi z \tag{11}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{y} - \mathbf{b}z = 0 \tag{12}$$

$$\mathbf{q}^T\mathbf{y} + \psi z = 1 \tag{13}$$

$$\mathbf{y} = z\mathbf{x} \tag{14}$$

$$\mathbf{y}, z \geq 0 \tag{15}$$

The normalized linear program of concern here is immediately obtained by setting ξ, ψ, and $\mathbf{b}$ equal to zero in the fractional program (equations (11)–(15)), that is,

$$\text{Minimize:} \quad z(\mathbf{p}^T\mathbf{x}) \tag{16}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} = \mathbf{0} \tag{17}$$

$$\mathbf{q}^T\mathbf{x} = \frac{1}{z} \tag{18}$$

in which external work in the original problem has simply been scaled by z.

Stochastic extensions: A hybrid model

A natural extension to the deterministic LP considers the plastic moment capacities (M_{pj}) and the external loads (e_i) as random variables.[11] In the initial formulation the M_{pj} are jointly normal, each e_i follows a normal distribution, and the M_{pj} are independent of the e_i. A stochastic programming model that explicitly uses these probabilistic descriptors is developed by first addressing the objective function.

The internal work objective function $\sum_{j=1}^{s} M_{pj}|\theta_j|$ is now a random variable. Assuming that the choice of θ_j does not affect the densities of the M_{pj}, the objective function is recast as

$$\text{Minimize:} \quad v \mid P\left[\sum_{j=1}^{s} M_{pj}|\theta_j| \geq v\right] = \alpha \tag{19}$$

That is, find the minimum collapse load factor v such that the probability of the actual (but unknown) collapse load factor $\sum_{j=1}^{s} M_{pj}|\theta_j|$ exceeding v is α. Following standard chance constrained programming (CCP) procedures[12,13] a solvable deterministic (i.e., nonstochastic) equivalent to equation (19) is obtained by minimizing

$$\sum_{j=1}^{s} E[M_{pj}]|\theta_j| + F^{-1}(1 - \alpha)\,(\theta^T V^M \theta)^{1/2} \tag{20}$$

where $E[M_{pj}]$ is the mean of M_{pj}, $F^{-1}(1 - \alpha)$ is the normal deviate corresponding to exceedence probability $(1 - \alpha)$, V^M is the variance-covariance matrix of the M_{pj}, and $(\theta^T V^M \theta)$ is the variance of $\sum_{j=1}^{s} M_{pj}|\theta_j|$. Note that the sign-unrestricted form of θ_j (i.e., $\theta_j^+ - \theta_j^-$) is actually used in the model but, to simplify the notation, is not shown above.

The random component in the constraint set is $\sum_{i=1}^{m} e_i t_i$. We cannot, however, use CCP techniques in this case because chance constraints cannot be developed for equalities. Alternatively, stochastic linear programming (SLP) methods[14] are used. These procedures involve generating random realizations of the

e_i from their respective densities, solving the associated optimization model, and repeating the process.[15-17] The model is therefore viewed as a *hybrid* in that it contains both CCP- and SLP-based components.

The model minimizes equation (20) subject to equations (6) and (7). The solution procedure consists of setting a value for $F^{-1}(1 - \alpha)$ and solving the overall model repeatedly, each solution corresponding to a different set of realizations for the e_i. A new value for $F^{-1}(1 - \alpha)$ is then selected, and the entire process repeated.

The portal frame shown in *Figure 1* was used to demonstrate the model. (This frame has seven critical sections and four elementary failure mechanisms. There are three additional linear combinations of these elementary mechanisms that are physically realistic.) The joint densities assumed for the moment capacities and the external loads are given in *Figure 1*.

Using the nonlinear optimization solver MINOS,[18] the hybrid optimization model was run 650 times (stochastic optimizations involving 50 randomly generated realizations of the e_i were performed for each of 13 selections of $F^{-1}(1 - \alpha) = 0.0, -0.5, -1.0, \ldots, -6.0$). This procedure was used to generate the contour plot of collapse load factors shown in *Figure 2*. The abscissa of *Figure 2* represents plastic moment capacity exceedence probability; as you move to the right, moment capacity realizations decrease. The ordinate of *Figure 2* represents external work (e_i) nonexceedence probability; as you move up, external work (i.e., load) realizations increase. (This nonexceedence probability was obtained from a normal distribution function that was fit to each set of 50 collapse load factors corresponding to the stochastic optimizations described above.)

A given structural design therefore tends to the conservative side when there is a high probability that the actual collapse load factor is *greater than* the optimization model's computed value. Thus relatively small moment capacity realizations (with high exceedence probabilities on the abscissa) coupled with large external work realizations (with high nonexceedence probabilities on the ordinate) lead to conservative (i.e., smaller) estimates of collapse load factor.

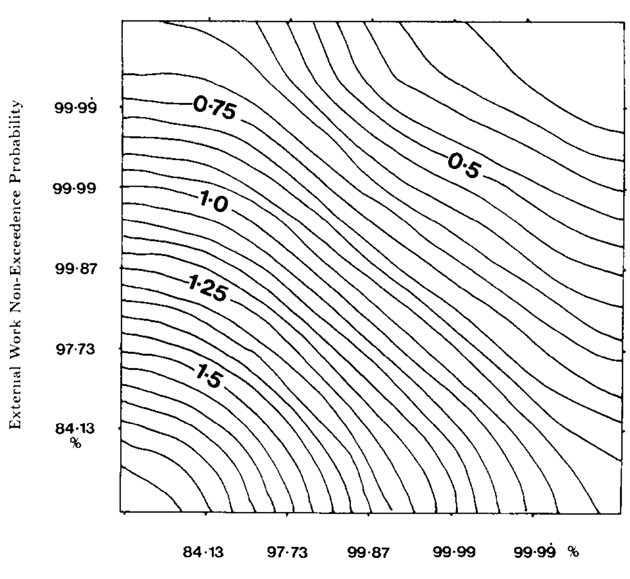

Figure 2. Hybrid model: collapse factor contours

Other scenarios were run in which the lateral and vertical external loads were assumed to follow Gumbel and gamma distributions, respectively. A drawback, however, is that because of the SLP component, the hybrid approach demands the execution of many nonlinear, nonseparable optimization models and can be rather laborious. Below, a different approach is attempted that employs only CCP methods and eliminates the need for a Monte Carlo-like treatment of external load stochasticity.

A purely CCP approach

The collapse load factor identification problem as originally formulated minimizes the ratio of internal work to external work, that is,

$$\textit{Minimize:} \quad \frac{\sum_{j=1}^{s} M_{pj}|\theta_j|}{\sum_{i=1}^{m} t_i e_i} \tag{21}$$

subject to the equilibrium conditions, equation (6). A new stochastic program was developed that has equation (21) as its objective function and equation (6) for the constraints. Note that both the numerator and the denominator of equation (21) are now random variables.

In the statistics literature, specifically on the subject of the distributions of quotients of random variables, Fieller[19] developed an expression that fit the general CCP concept desired. Fieller provided a method to calculate the probability α that a quotient of normal random variables (N/D) exceeds a specified number v, that is,

$$P\left[\frac{N}{D} \geq v\right] = \alpha \tag{22}$$

This probability $\left(P\left[z = \frac{N}{D} \geq v\right] \equiv \{1 - F_z(v)\}\right)$ is given by

$$\{1 - F_z(v)\} = \int_h^\infty \int_k^\infty B(\rho)\,dx\,dy + \int_{-h}^\infty \int_{-k}^\infty B(\rho)\,dx\,dy \tag{23}$$

where N and D are the numerator and denominator of equation (21), respectively, $B(\rho)$ is the standardized bivariate normal, and

$$h = \frac{E[D]}{\sigma_D} \tag{24}$$

$$k = \frac{E[N] - vE[D]}{[\sigma_N^2 - 2\sigma_{ND}v + \sigma_D^2 v^2]^{1/2}} \tag{25}$$

$$\rho = \frac{\sigma_{ND} - v\sigma_D^2}{\sigma_D[\sigma_N^2 - 2\sigma_{ND}v + \sigma_D^2 v^2]} \quad (26)$$

For the case at hand,

$$E[D] = \sum_{i=1}^{m} E[e_i]t_i \quad (27)$$

$$\sigma_D^2 = (t^T V^e t) \quad (28)$$

$$E[N] = \sum_{j=1}^{s} E[M_{pj}]|\theta_j| \quad (29)$$

$$\sigma_N^2 = (\theta^T V^M \theta) \quad (30)$$

The covariance between moment capacities and external loads (σ_{ND}) is zero for these analyses. The solution procedure involves first selecting a collapse load factor v and subsequently finding the minimum probability that this value will be exceeded (that is, find the failure mode with the minimum reliability). Again the situation wherein the actual but unknown collapse load factor is greater than the specified level v is interpreted as conservative or risk-adverse. The probability that this situation will occur is α.

The portal frame model (minimize (23) subject to (6) and (7)) was again solved with MINOS (see the Appendix for a description of the gradients of equation (23)) with execution times on a Vaxstation 3500 that were consistently less than one second. The optimization problem is nonconvex, with local optima that are starting point–dependent and failure mode–specific. For example, consider *Figure 3*, which depicts the example problem in so-called load space $[L_1, L_2]$.

In load space the means of the lateral and vertical loads $(\overline{L_1}, \overline{L_2})$ occur at the beam/combined failure transition line. When the optimization model is solved by using a starting point that represents beam failure, the solid tradeoff curve shown in *Figure 4* is obtained. Note that there is a continuum of equivalent reliability-collapse load factor pairs for that given structure. In other words, for this design there is a 99.999% chance

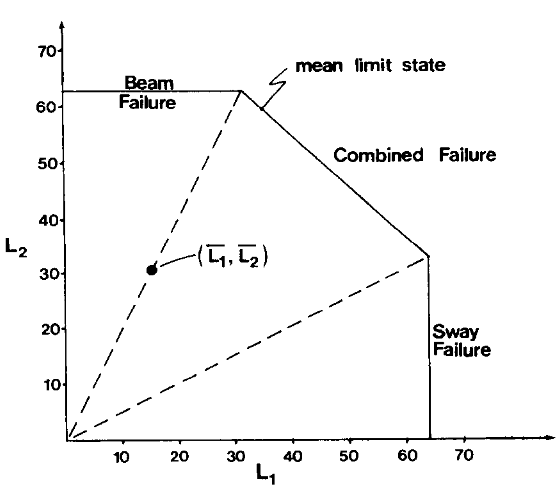

Figure 3. Load space diagram for the portal frame

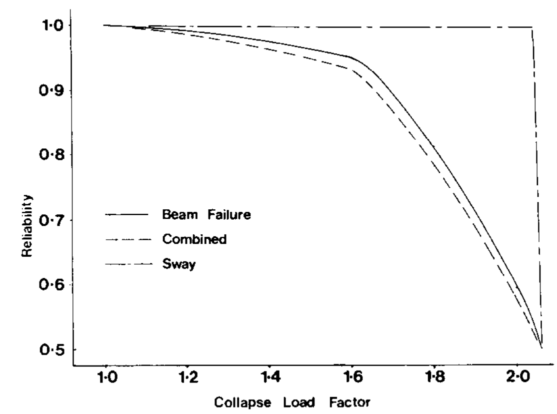

Figure 4. Tradeoff of collapse load factor versus reliability

that the actual but unknown collapse load factor will be greater than 1. For this same design there is also (and equivalently) an 81% chance that the actual collapse load factor will be greater than 1.8, and so on. Furthermore, reliability drops to exactly 50% when the collapse load factor takes on a value that, in load space, scales the original lateral and vertical load means to positions directly on the mean limit state. Also shown in *Figure 4* are the tradeoffs obtained when the combined and the sway starting points are used.

Figure 4 also clearly shows the local optimality of the model. If a sway starting point is used, sway failures result, and the associated (incorrect) reliability with respect to sway failure remains near unity. This should not occur because combined failure represents the global optimum. Owing to nonconvexity of the objective function, however, the optimization model, given a sway starting point, terminates on a projection of the sway limit state outside of the region delineated by the load axes and the mean limit state (inside these boundaries is the safe or no-failure region). A sway failure on this projection will, in reality, not occur because combined failure will have already occurred.

In the context of the original *critical* collapse load factor identification problem, these three tradeoffs must be compared, for example, for a given reliability the tradeoff curve that yields the minimum collapse load factor identifies both the critical factor and the failure mode associated with that factor. The critical factor is thus seen to always be associated with the combined failure mechanism for the example problem. Also, the tradeoffs shown in *Figure 4* would seem to bear out our intuitive expectations in view of the relative position of the external load means in the load space diagram of *Figure 3*.

Here it is useful to compare the new stochastic model with traditional approaches to estimating structural reliability. In traditional methods of analysis, kinematically admissable failure modes are obtained by forming new modes from linear combinations of the elementary failure mechanisms. For each failure mechanism the internal and external work (N and D) can be calculated, and the reliability for each enumerated mode can be calculated by using equations (22) and (23). Thus failure modes must be enumerated either by brute force or by some heuristic to obtain a set of modes for which the reliability is calculated. Then the mode with the lowest reliability is considered the critical mode.

Objective function simplification

As was mentioned in the previous section, the objective of the stochastic version of the frame analysis is to find the least reliable collapse mode. Rather than directly using the CCP approach based on Fieller's results, a simplified model is possible. Equation (22) can be expressed as $P[N - \nu D \geq 0]$ (for $D \geq 0$, which is physically realistic), and assuming that N and D are independent, a new random variable, $Z = N - \nu D$, can be introduced with $m_Z = m_N - \nu m_D$ and $\sigma_Z^2 = \sigma_N^2 + \nu^2\sigma_D^2$. With an assumption of normality for the loads and moment capacities the objective function can now be expressed as

Minimize:

$$\alpha = P[Z \geq 0] = \int_0^{\infty} f_Z\, dz = \Phi\left(\frac{m_Z}{\sigma_Z}\right) = \Phi(\beta) \quad (31)$$

Since the cumulative distribution, $\Phi(-)$, is monotonic, an equivalent formulation to the problem is

$$\textit{Minimize:} \quad \beta = \frac{m_N - \nu m_D}{\sqrt{\sigma_N^2 + \nu^2\sigma_D^2}} \quad (32)$$

$$\textit{Subject to:} \quad \theta_j - \sum_{i=1}^{m} t_i\theta_{ij} = 0$$

$$j = 1, \ldots, s \quad (33)$$

It should be noted that if $\nu = 1$, β coincides with the usual concept of the reliability index.[20] This form of the problem was investigated by Ishikawa and Iizuka,[21] although the connection to CCP methods was not explored. Equation (32) is nonlinear but is computationally more attractive than equation (31). The objective function was again found to be nonconvex.

Nonconvexity

Using the simplified model described above, we now demonstrate in more detail the nonconvexity of the mathematical program set forth by equations (32) and (33). The constraints in equation (33) can be solved for the rotations, θ_j, and these decision variables can be eliminated from the objective function, β, such that the math program is only a function of the participation factors, t_i. For the same single-bay, single-story structure shown earlier, β can be written in terms of the four participation factors, t_1 through t_4. By holding $t_2 = 1$ and $t_4 = -1$ the unconstrained objective function surface can be plotted and is shown in *Figures 5* and *6*. The plotted surface is the value of β over a preselected range of t_1 and t_3. The point at $t_1 = 0$ and $t_3 = 1$ is a local optimum (beam mode). The point at $t_1 = 1$ and $t_3 = 1$ is the global optimum (combination

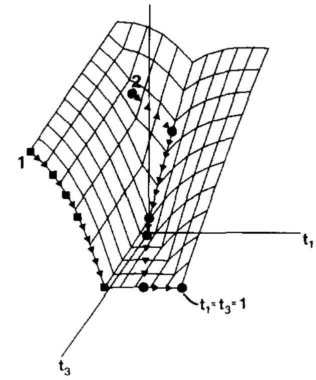

Figure 5. Objective response surface: starting points 1 and 2

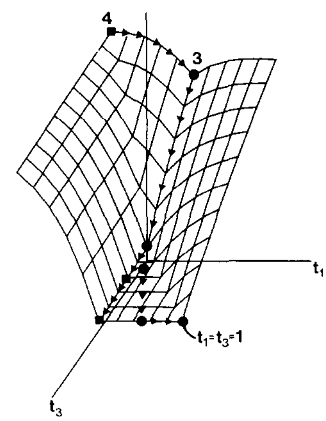

Figure 6. Objective response surface: starting points 3 and 4

mode). There is a slight rise of the surface along the line $t_3 = 1$.

Figures 5 and *6* show the effect of starting point on the final solution found by MINOS. In *Figure 5*, two starting points and the paths to their final destination are shown (the numbers indicate the starting point). The path to solution stays on the objective function surface (that is, the values of t_2 and t_4 remain constant throughout the optimization). Starting point 1 leads to the local (beam) solution, and starting point 2 leads to the (globally optimal) combination failure mode. *Figure 6* shows more nonintuitive behavior. Both starting points (3 and 4) lead to the point $t_1 = t_3 = 0$, but then the paths diverge, one solution ending in beam failure and the other in combined failure.

Conclusions

Three models were developed to extend the traditional deterministic collapse load factor identification problem to stochastic forms. The first, termed a hybrid, retained the basic LP structure in that the external work relation was normalized and set as a constraint. This model, involving both chance constrained and stochastic linear programming concepts was solved by using MINOS and proved to be an acceptable but laborious method for examining the effects of load and moment capacity stochasticity on critical collapse load factor. A new model that retained the original quotient form of the objective function (the ratio of internal to external work) was developed by using a result from Fieller. The subsequent optimization model represents a more complex formulation but nonetheless solves very rapidly. A principal attribute of this approach consists of its ability to explicitly incorporate input joint densities for both external loads and moment capacities into a model that does not require a Monte Carlo simulation or optimization task at any stage. Finally, a third model was described that represents, for the applications at hand, a simplification of the Fieller-based CCP approach.

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Appendix

In addition to evaluating the objective function of the purely CCP model, that is,

$$\{1 - F_z(v)\} = G_z(v) = \int_h^\infty \int_k^\infty B(\rho)\, dx\, dy + \int_{-h}^\infty \int_{-k}^\infty B(\rho)\, dx\, dy \tag{A1}$$

MINOS (or any reduced gradient technique) also requires gradients of this expression with respect to the decision variables. These gradients can be estimated by MINOS using finite difference approximations; but given the complexity of the function being evaluated, a much more reliable and efficient approach is to provide analytic gradients, and that is the approach taken here. Through a procedure that involves the application of Liebniz's theorem for differentiating integrals[22] and a reduction formula for multivariate integrals due to Plackett,[23] these gradients can be shown as

$$\begin{aligned}\frac{\partial G(x)}{\partial x_j} = \frac{\partial h(x)}{\partial x_j}[\{\phi(-h(x)) + \phi(h(x))\}\cdot F(w_1(x)) - \phi(h(x))] + \frac{\partial k(x)}{\partial x_j}[\{\phi(-k(x)) + \phi(k(x))\}\cdot F(w_2(x)) - \phi(k(x))] \\ + \frac{\partial \rho(x)}{\partial x_j}\{B(\rho(x), k(x), h(x))\} + \frac{\partial \rho(x)}{\partial x_j}\{B(\rho(x), -k(x), -h(x))\}\end{aligned} \tag{A2}$$

where x_j represents our decision variables θ_j^+, θ_j^-, t_i^+, t_i^-, v, and in addition to the definitions given in the text,

$$\phi(\cdot) = N(0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-z^2}{2}\right\} \tag{A3}$$

$$F(\cdot) = \int_{-\infty}^{z} \phi(y)\, dy \tag{A4}$$

$$w_1(x) = \frac{k(x) - \rho(x)h(x)}{\sqrt{1 - \rho(x)^2}} \tag{A5}$$

$$w_2(x) = \frac{h(x) - \rho(x)k(x)}{\sqrt{1 - \rho(x)^2}} \tag{A6}$$